The purpose of this short note is to find estimates on the number of zeros of solutions of the equation

$$u''(x) + \frac{C}{x^{2\alpha}} u(x) = 0\,, \tag{1}$$

for $x \in [1, \infty)$ and $\alpha > 1$. We will establish that the number of zeros $N = N(C, \alpha)$ satisfies

$$N = \mathrm{O}\left(\frac{\sqrt{C}}{\alpha - 1}\right). \tag{2}$$

When $\alpha = 1$ it is known that solutions of (1) have finitely (in fact, at most one) or infinitely many zeros depending on whether $C \leq 1/4$ or $C > 1/4$.

Let x_f be defined by the equation

$$\frac{C}{x_f^{2\alpha}} = \frac{1/4}{x_f^2}\,,$$

that is

$$x_f^{\alpha-1} = 2\sqrt{C}\,.$$

It follows that $C/x^{2\alpha} \leq (1/4)(1/x^2)$ exactly for $x \geq x_f$, and thus by the comments above and Sturm comparison, the number of zeros of solutions of (1) in the interval $[x_f, \infty)$ is at most one. With this, let us define the recurrence relation

$$x_{k+1} = x_k + \epsilon x_k^{\alpha}\,, \tag{3}$$

where $\epsilon = \pi/\sqrt{C}$. Because in the interval $I_k = [x_k, x_{k+1}]$ one has $C/x^{2\alpha} \leq C/x_k^{2\alpha}$, it follows from Sturm comparison that the number of zeros contained in I_k of any solution of (1) cannot exceed 1. Hence we need to estimate the number m of iterations required to reach the point x_f under the recurrence relation (3) starting with $x_0 = 1$. To do this we do an area comparison. Let us consider the function $y = 1/(\epsilon x^{\alpha})$. The area under its graph in the interval I_k, say A_k, satisfies

$$\frac{1}{\epsilon}\frac{x_{k+1} - x_k}{x_{k+1}^{\alpha}} \leq A_k \leq \frac{1}{\epsilon}\frac{x_{k+1} - x_k}{x_k^{\alpha}} = 1\,. \tag{4}$$

On the other hand,

$$\frac{x_{k+1}}{x_k} = 1 + \epsilon x_k^{\alpha-1} \leq 1 + \epsilon x_f^{\alpha-1} = 1 + 2\pi\,,$$

which shows that the number m and of iterations, and thus N, are bounded above by

$$(1+2\pi)^{\alpha} \int_1^{x_f} \frac{dx}{\epsilon x^{\alpha}} = \frac{(1+2\pi)^{\alpha}}{\epsilon(\alpha-1)}\left(1 - \frac{1}{x_f^{\alpha-1}}\right) \leq \frac{(1+2\pi)^{\alpha}\sqrt{C}}{\pi(\alpha-1)}\,. \tag{5}$$

In order to get a lower for N we argue as follows. Let $0 < r_0 < 1$ be fixed. We will define $0 < r_{k+1} < r_k$ recursively in such a way that any solution of (1) is guaranteed to have at least one zero in the interval $J_k = [r_{k+1}x_f, r_k x_f]$. Suppose $0 < r_k < 1$ is defined. It is easy to see that for $1 \leq x \leq r_k x_f$ one has

$$\frac{C}{x^{2\alpha}} \geq \left(\frac{1 + a_k}{4}\right)\frac{1}{x^2}\,, \tag{6}$$

where a_k is given by the equation

$$r_k^{\alpha-1} = \frac{1}{\sqrt{1 + a_k}}\,. \tag{7}$$

On $[1, r_k x_f]$ we compare equation (1) with

$$v'' + \left(\frac{1+a_k}{4}\right)\frac{1}{x^2}v = 0\,, \tag{8}$$

the solutions of which are given by linear combination of

$$\sqrt{x}\sin\left(\frac{1}{2}\sqrt{a_k}\log x\right) \quad \text{and} \quad \sqrt{x}\cos\left(\frac{1}{2}\sqrt{a_k}\log x\right)\,.$$

It follows that any solution of (8), and thus of (1), will have a zero in the interval $J_k = [r_{k+1}x_f, r_k x_f]$ provided

$$\log r_k - \log r_{k+1} = \frac{2\pi}{\sqrt{a_k}}\,. \tag{9}$$

We use (9) to define the $r_k's$ recursively. Notice from (7) that the $a_k's$ will be increasing.

We now need to estimate how many iterations are required to bring $r_k x_f$ for the first time below the value 1, that is, roughly when $r_k = 1/x_f$. We will do this again by resorting to integrals. Let $s_k = -\log r_k$. Then (9) becomes

$$s_{k+1} - s_k = \frac{2\pi}{\sqrt{e^{2(\alpha-1)s_k} - 1}}\,. \tag{10}$$

We need to estimate how many steps are needed, roughly, to make $s_k = \log x_f$. Consider the function

$$t = \frac{1}{2\pi}\sqrt{e^{2(\alpha-1)s_k} - 1}\,.$$

On each interval J_k the area B_k under its graph satisfies

$$1 = \frac{1}{2\pi}\sqrt{e^{2(\alpha-1)s_k} - 1}\,(s_{k+1} - s_k) \;\le B_k \;\le\; \frac{1}{2\pi}\sqrt{e^{2(\alpha-1)s_{k+1}} - 1}\,(s_{k+1} - s_k)\,. \tag{11}$$

But

$$\frac{e^{(\alpha-1)s_{k+1}}}{e^{(\alpha-1)s_k}} = e^{(\alpha-1)(s_{k+1}-s_k)} = \left(\frac{r_k}{r_{k+1}}\right)^{\alpha-1} = e^{\frac{2\pi(\alpha-1)}{\sqrt{a_k}}} \le e^{\frac{2\pi(\alpha-1)}{\sqrt{a_0}}}\,,$$

and therefore

$$\frac{1}{2\pi}\int_{s_0}^{\log x_f} \sqrt{e^{2(\alpha-1)s} - 1}\,ds$$

is comparable to the number of iterations to be determined. For large C and thus large x_f this integral is comparable to

$$\frac{1}{2\pi}\int_{s_0}^{\log x_f} e^{(\alpha-1)s}\,ds\,,$$

which is easily computed and found to be less than $\sqrt{C}/(\alpha-1)$.